On Spectral Densities

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***Abstract*— Spectral density is a measure on the average power of a signal as a function of frequency. There are two different definitions used in the literature. This paper serves to underline the difference between them. We also give the origin of spectral density.**

Warning: This is not an introduction to spectral density. If the subject is completely unfamiliar I'd advise reading another source. For example chapter 4 in [1] or chapter 7 in [2].

I. Definition of Spectral Density

There are two different definitions of spectral density used in the literature. They differ by a factor of two. The one used in signal processing books, like [3], is

$$S_{x1}(f) = \int_{-\infty}^{\infty} R_{x1}(\tau) e^{-j\omega\tau} d\tau \tag{1}$$

And the one often used in books about noise, like [4], is

$$S_{x2}(f) = 2\int_{-\infty}^{\infty} R_{x2}(\tau) e^{-j\omega\tau} d\tau \tag{2}$$

In both cases $R_{xi}(\tau)$ is the auto-correlation function defined as

$$R_{xi}(\tau) = \overline{x_i(t)x_i(t+\tau)} \tag{3}$$

As we can plainly see

$$S_{x1}(f) \neq S_{x2}(f) \tag{4}$$

, there is no way these two can be made equal if

$$R_{x1}(\tau) = R_{x2}(\tau) \tag{5}$$

This is ok, there is no problem having two different definitions for two different functions. In reality $S_{x1}(f)$ and $S_{x2}(f)$ are different functions of frequency, and we could say that

$$S_{x2}(f) = 2S_{x1}(f) \tag{6}$$

if (5) is true.

II. Sources of Confusion

The problem with spectral density arises when reading literature from different communities, for example [3] and [4] where $S_x(f)$ is used for both $S_{x1}(f)$ and $S_{x2}(f)$. When I started investigating spectral densities this lead me to believe that different sources defined the same measure “spectral density” in two different ways. The more sources I investigated the more unsure I was about which of the two definitions that was correct. After months of searching (not actively, but sporadicly) I eventually found the original source of the definition of spectral density [5]. Having the original source helped, but I still don't know when the original definition split into (1) and (2). However, I'm pretty sure the it's just a matter of convenience. To see why (2) is the most common among sources concerning noise we look at the inverse Fourier Transform. By the way, if you had not noticed yet, (1) says that *Spectral density is the Fourier Transform of the Auto-Correlation function*. The inverse Fourier Transform of (1) is

$$R_{x1}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{x1}(f) e^{j\omega\tau} dw = \int_{-\infty}^{\infty} S_{x1}(f) e^{j\omega\tau} df \tag{7}$$

,since $dw = df dw/df = 2\pi df$. And for (2)

$$R_{x2}(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} S_{x2}(f) e^{jw\tau} df \tag{8}$$

Before we proceed lets get rid of the e's. We know that $e^{\alpha} = \cos\alpha + j\sin\alpha$. So we could rewrite (1) as

$$S_{x1}(f) = \int_{-\infty}^{\infty} R_{x1}(\tau)[\cos(\omega\tau) + j\sin(\omega\tau)] d\tau \tag{9}$$

and it turns out that since $R_{x1}(\tau)$ is an even function we can drop the $j\sin\omega\tau$ term. $S_{x1}(f)$ is also an even function since the Fourier Transform of an even function is even.

The definitions then become

$$\begin{aligned} S_{x1}(f) &= \int_{-\infty}^{\infty} R_{x1}(\tau)\cos(\omega\tau) d\tau \\ R_{x1}(\tau) &= \int_{-\infty}^{\infty} S_{x1}(f)\cos(\omega\tau) df \end{aligned} \tag{10}$$

and

$$\begin{aligned} S_{x2}(f) &= 2\int_{-\infty}^{\infty} R_{x2}(\tau)\cos(\omega\tau) d\tau \\ R_{x2}(\tau) &= \frac{1}{2}\int_{-\infty}^{\infty} S_{x2}(f)\cos(\omega\tau) df \end{aligned} \tag{11}$$

We can rewrite $R_{x2}(\tau)$ as

$$R_{x2}(\tau) = \overline{x_2(t)x_2(t+\tau)} = \int_{0}^{\infty} S_{x2}(f)\cos(\omega\tau) df \tag{12}$$

and if $\tau = 0$

$$\overline{x_2^2(t)} = \int_0^{\infty} S_{x2}(f) df \tag{13}$$

So using spectral density definition (2) we see that average power (mean square value of $x_2(t)$) is equal to the integral from 0 to infinity of the spectral density. If we use (1) average power would be

$$\overline{x_1^2(t)} = 2\int_0^{\infty} S_{x1}(f) df \tag{14}$$

But if $R_{x1}(\tau) = R_{x2}(\tau)$ then

$$\overline{x_2^2(t)} = \overline{x_1^2(t)} \tag{15}$$

even though $S_{x1}(f) \neq S_{x2}(f)$.

Definition (1) is called the two-sided spectral density and (2) is called the one-sided spectral density.

III. Example: Thermal Noise

The spectral density of thermal noise in electronic circuit should be known to anyone that has studied analog electronics. We normally define the voltage spectral density of thermal noise as

$$S_{th}(f) = 4kTR \tag{16}$$

where k is Boltzmann's constant, T the temperature in Kelvin and R the resistance. But (16) is the spectral density when it is defined as in (2). If we were to use (1) then the spectral density of thermal noise would be

$$S_{th}(f) = 2kTR \tag{17}$$

Both these spectral densities would give the same average power value if we use the inverse Fourier Transform of (1) and (2).[1]

IV. Einstein: The source

In his 1914 paper [5] Albert Einstein described, supposedly for the first time, the auto-correlation function and what we have come to know as the spectral density. He defined the auto-correlation function as

$$\mathfrak{M}(\Delta) = \overline{F(t)F(t+\Delta)} \tag{18}$$

and the intensity (spectral density) as

$$I(\theta) = \int_0^T \mathfrak{M}(\Delta)\cos(\pi\frac{\Delta}{\theta})d\Delta \tag{19}$$

,where the period $\theta = T/n$ and T is a very large value. The paper is very short, only 1 page, but it is worth reading. Note that (1) is often referred to as the *Wiener-Khintchine* theorem.

References

[1] D. Johns and K. Martin, *Analog Integrated Circuit Design*. John Wiley & Sons, Inc., 1997.

[2] B. Razavi, *Design of Analog CMOS Integrated Circuits*. McGraw-Hill, 2001.

[3] R. M. Gray and L. D. Davisson, *An Introduction to Statistical Signal Processing*. Cambridge University Press, 2004.

[4] A. V. D. Ziel, *Noise in Solid State Devices and Circuits*. John Wiley & Sons Inc, 1986.

[5] A. Einstein, "Method for the determinination of the statistical values of observations concerning quantities subject to irregular fluctuations," *IEEE ASSP Mag.*, vol. October, 1987.

[1]Note that if you calculate the average power of $S_{th}(f)$ you'll get infinity. You have to include the bandwidth of the circuit you are considering for average power to have a finite value.