Introduction to the Mathematics of Noise Sources

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***Abstract*— This is a compilation of different books [1]–[3] and their introduction to noise analysis of electronic circuits.**

I. Noise

Noise is a phenomena that occurs in all electronic circuits. It places a lower limit on the smallest signal we can use. Many now have super audio compact disc (SACD) players with 24bit converters, 24 bits is around $2^{24} = 16.78$ Million different levels. If 5V is the maximum voltage, the minimum would have to be $\frac{5V}{2^{24}} \approx 298nV$. That level is roughly equivalent to the noise in a 50 Ohm resistor with a bandwith of 96kHz. There exist an equation that relates number of bits to signal to noise ratio [2], the equation specifies that $SNR = 6.02 * Bits + 1.76 = 146.24dB$. As of 12.2005 the best digital to analog converter (DAC) that Analog Devices (a very big semiconductor company) has is a DAC with 120dB SNR, that equals around $Bits = (120 - 1.76)/6.02 = 19.64$. In other words, the last four bits of your SACD player is probably noise!

II. Statistics

The mean of a signal x(t) is defined as

$$\overline{x(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)dt \qquad (1)$$

The mean square of x(t) defined as

$$\overline{x^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)dt \qquad (2)$$

The variance of x(t) defined as

$$\sigma^2 = \overline{x^2(t)} - \overline{x(t)}^2 \qquad (3)$$

For a signals with a mean of zero the variance is equal to the mean square. The auto-correlation of x(t) is defined as

$$\begin{aligned} R_x(\tau) &= \overline{x(t)x(t+\tau)} \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau)dt \qquad (4) \end{aligned}$$

III. Average Power

Average power is defined for a continuous system as (5) and for discrete samples it can be defined as (6). P_{av} usually has the unit A^2 or V^2, so we have to multiply/devide by the impedance to get the power in Watts. To get Volts and Amperes we use the root-mean-square (RMS) value which is defined as $\sqrt{P_{av}}$.

$$P_{av} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)dt \qquad (5)$$

$$P_{av} = \frac{1}{N} \sum_{i=0}^{N} x^2(i) \qquad (6)$$

If x(t) has a mean of zero then, according to (3), P_{av} is equal to the variance of x(t).

Many different notations are used to denote average power and RMS value of voltage or current, some of them are listed in Table I and Table II. Notation can be a confusing thing, it changes from book to book and makes expressions look different. It is important to realize that it does not matter how you write average power and RMS value. If you want you can invent your own notation for average power and RMS value. However, if you are presenting your calculations to other people it is convenient if they understand what you have written. In the remainder of this paper we will use $\overline{e_n^2}$ for average power when we talk about voltage noise source and $\overline{i_n^2}$ for average power when we talk about current noise source. The n subscript is used to identify different sources and can be whatever.

TABLE I
Notations for average power

Voltage	Current
V_{rms}^2	I_{rms}^2
$\overline{V_n^2}$	$\overline{I_n^2}$
$\overline{v_n^2}$	$\overline{i_n^2}$

TABLE II
Notations for RMS

Voltage	Current
V_{rms}	I_{rms}
$\sqrt{\overline{V_n^2}}$	$\sqrt{\overline{I_n^2}}$
$\sqrt{\overline{v_n^2}}$	$\sqrt{\overline{i_n^2}}$

IV. Noise Spectrum

With random noise it is useful to relate the average power to frequency. We call this Power Spectral Density (PSD). A PSD plots how much power a signal carries at each frequency. In literature $S_x(f)$ is often used to denote the PSD. In the same way that we use V^2 as unit of average power, the unit of the PSD is $\frac{V^2}{Hz}$ for voltage and $\frac{A^2}{Hz}$ current. The root spectral density is defined as $\sqrt{S_x(f)}$ and has unit $\frac{V}{\sqrt{Hz}}$ for voltage and $\frac{I}{\sqrt{Hz}}$ for current.

The power spectral density is defined as two times the Fourier transform of the auto-correlation function [1]

$$S_x(f) = 2 \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau \qquad (7)$$

This can also be written as

$$\begin{aligned} S_x(f) &= 2\left[\int_{-\infty}^{\infty} R_x(\tau)\cos(\omega\tau)d\tau - \int_{-\infty}^{\infty} R_x(\tau)j\sin(\omega\tau)d\tau\right] \\ &= 2\left[\int_{-\infty}^{0} R_x(\tau)\cos(\omega\tau)d\tau + \int_{0}^{\infty} R_x(\tau)\cos(\omega\tau)d\tau\right] \\ &- 2j\left[\int_{-\infty}^{0} R_x(\tau)\sin(\omega\tau)d\tau + \int_{0}^{\infty} R_x(\tau)\sin(\omega\tau)d\tau\right] \\ &= 4\int_{0}^{\infty} R_x(\tau)\cos(\omega\tau)d\tau \\ &- 2j\left[-\int_{0}^{\infty} R_x(\tau)\sin(\omega\tau)d\tau + \int_{0}^{\infty} R_x(\tau)\sin(\omega\tau)d\tau\right] \\ &= 4\int_{0}^{\infty} R_x(\tau)\cos(\omega\tau)d\tau \end{aligned} \tag{8}$$

, since $e^{-j\omega\tau} = \cos(\omega\tau) - j\sin(\omega\tau)$, $R_x(\tau)$ and $\cos(\omega\tau)$ are symmetric around $\tau = 0$ while $\sin(\omega\tau)$ is asymmetric around $\tau = 0$.

The inverse of power spectral density is defined as

$$R_x(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} S_x(f)e^{j2\pi f\tau}df = \int_{0}^{\infty} S_x(f)\cos(\omega\tau)df \tag{9}$$

If we set $\tau = 0$ we get

$$\overline{x^2(t)} = \int_{0}^{\infty} S_x(f)df \tag{10}$$

which means we can easily calculate the average power if we know the power spectral density. As we will see later it is common to express noise sources in PSD form.

Another very useful theorem when working with noise in the frequency domain is this

$$S_y(f) = S_x(f)|H(f)|^2 \tag{11}$$

, where $S_y(f)$ is the output power spectral density, $S_x(f)$ is the input power spectral density and $H(f)$ is the transfer function of a time-invariant linear system.

If we insert (11) into (10), with $S_x(f) = a\,constant = D_v$ we get

$$\overline{x^2(t)} = \int S_y(f)df = D_v\int |H(f)|^2 df = D_v f_x \tag{12}$$

, where f_x is what we call the noise bandwidth. For a single time constant RC network the noise bandwidth is equal to

$$f_x = \frac{\pi f_0}{2} = \frac{1}{4RC} \tag{13}$$

where f_x is the noise bandwidth and f_0 is the 3dB frequency.

We haven't told you this yet, but thermal noise is white and white means that the power spectral density is flat (constant over all frequencies). If $S_x(f)$ is our thermal noise source and $H(f)$ is a standard low pass filter, then equation (11) tells us that the output spectral density will be shaped by $H(f)$. At frequencies above the f_x in $H(f)$ we expect the root power spectral density to fall by 20dB per decade.

V. Probability Distribution

Theorem 1 (Central limit theorem): The sum of n independent random variables subjected to the same distribution will always approach a normal distribution curve as n increases.

This is a neat theorem, it explains why many noise sources we encounter in the real world are white.[1] Take thermal noise for example, it is generated by random motion of carriers in materials. If we look at a single electron moving through the material the probability distribution might not be Gaussian. But summing probability distribution of the random movments with a large number of electrons will give us a Gaussian distribution, thus thermal noise is white.

VI. PSD of a white noise source

If we have a true random process with Gaussian distribution we know that the autocorrelation function only has a value for $\tau = 0$. From equation (4) we have that

$$\begin{aligned} R_x(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t-\tau)dt \\ &= \left[\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)dt\right]\delta(\tau) \\ &= \overline{x^2(t)}\delta(\tau) \end{aligned} \tag{14}$$

The reason being that in a true random process $x(t)$ is uncorrelated with $x(t+\tau)$ where τ is an integer. If we use equation (7) we see that

$$\begin{aligned} S_x(f) &= 2\int_{-\infty}^{\infty} \overline{x^2(t)}\delta(\tau)e^{-j2\pi f\tau}d\tau \\ &= 2\overline{x^2(t)}\int_{-\infty}^{\infty}\delta(\tau)e^{-j2\pi f\tau}d\tau \\ &= 2\overline{x^2(t)} \end{aligned} \tag{15}$$

, since

$$\int \delta(\tau)e^{-j2\pi f\tau}d\tau = e^0 = 1 \tag{16}$$

This means that the power spectral density of a white noise source is flat, or in other words, the same for all frequencies.

VII. Summing noise sources

Summing noise sources is usually trivial, but we need to know why and when it is not. We if we write the time dependant noise signals as

$$v_{tot}^2(t) = (v_1(t)+v_2(t))^2 = v_1^2(t) + 2v_1(t)v_2(t) + v_2^2(t) \tag{17}$$

[1] Gaussian distribution = normal distribution. Noise sources with Gaussian distribution are called white

The average power is defined as

$$\begin{aligned}\overline{e_{tot}^2} &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} v_{tot}^2(t)dt \\ &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} v_1^2(t)dt \\ &+ \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} v_2^2(t)dt \\ &+ \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} 2v_1(t)v_2(t)dt \\ &= \overline{e_1^2} + \overline{e_2^2} + \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} 2v_1(t)v_2(t)dt \qquad (18)\end{aligned}$$

If $\overline{e_1^2}$ and $\overline{e_2^2}$ are uncorrelated noise sources we can skip the last term in (18) and just write

$$\overline{e_{tot}^2} = \overline{e_1^2} + \overline{e_2^2} \qquad (19)$$

Most natural noise sources are uncorrelated.

VIII. Signal to Noise Ratios

Signal to Noise Ratio (SNR) is a common method to specify the relation between signal power and noise power in linear systems. It is defined as

$$\begin{aligned}SNR &= 10\log\left(\frac{Signal\ power}{Noise\ power}\right) \\ &= 10\log\left(\frac{\overline{v_{sig}^2}}{\overline{e_n^2}}\right) \\ &= 20\log\left(\frac{v_{rms}}{\sqrt{\overline{e_n^2}}}\right) \qquad (20)\end{aligned}$$

Another useful ratio is Signal to Noise and Distortion (SNDR), since most real systems exibit non-linearities it is useful to include distortion in the ratio. One can calculate SNR and SNDR in many ways. If we don't know the expression for $\overline{e_n^2}$ we can do a FFT of our output signal. From this FFT we sum spectral components except at the signal frequency to get noise and distortion. SNR is normally calculated as

$$SNR = 10\log\left(\frac{Signal\ power}{Noise\ power\ -\ 6\ first\ harmonics}\right) \qquad (21)$$

And SNDR is calculated as

$$SNDR = 10\log\left(\frac{Signal\ power}{Noise\ power}\right) \qquad (22)$$

IX. Noise figure and Friis formula

Noise factor is a measure on the noise performance of a system. It is defined as

$$F = \frac{\overline{v_o^2}}{source\ contribution\ to\ \overline{v_o^2}} \qquad (23)$$

where $\overline{v_o^2}$ is the total output noise.

The noise figure is defined as (noise factor in dB)

$$NF = 10\log(F) \qquad (24)$$

The noise factor can also be defined as

$$F = \frac{SNR_{input}}{SNR_{output}} \qquad (25)$$

This brings us right into what is known as Friis formula. If we have a multistage system, for example several amplifiers in cascade, the total noise figure of the system is defined as

$$F = 1 + F_1 - 1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + ... \qquad (26)$$

Here F_i is the noise figures of the individual stages and G_i is the available gain of each stage. This can be rewritten as

$$F = F_1 + \sum_{i=1}^{N}\frac{F_{i+1} - 1}{\prod_{k=1}^{i-1} G_i} \qquad (27)$$

Friiss formula tells us that it is the noise in the first stage that is the most important if G_1 is large. We could say that in a system it is important to amplify the noise as early as possible!

X. Conclusion

We have looked at the properties of noise in time domain and frequency domain. The equations in this paper are useful tools when dealing with noise sources.

References

[1] A. V. D. Ziel, *Noise in Solid State Devices and Circuits.* John Wiley & Sons Inc, 1986.
[2] D. Johns and K. Martin, *Analog Integrated Circuit Design.* John Wiley & Sons, Inc., 1997.
[3] B. Razavi, *Design of Analog CMOS Integrated Circuits.* McGraw-Hill, 2001.